01 Nov 1991

Analysis of Baseband Equivalent Noise in a First-Order Correlation Loop Utilizing Filtered Pseudonoise Signals

Kurt Louis Kosbar
Missouri University of Science and Technology, kosbar@mst.edu

Christopher J. Scholten

Recommended Citation

K. L. Kosbar and C. J. Scholten, "Analysis of Baseband Equivalent Noise in a First-Order Correlation Loop Utilizing Filtered Pseudonoise Signals," *MILCOM 91: Conference Record : Military Communications in a Changing World*, vol. 1, pp. 355-359, Institute of Electrical and Electronics Engineers (IEEE), Nov 1991.
The definitive version is available at https://doi.org/10.1109/MILCOM.1991.258272

ANALYSIS OF BASEBAND EQUIVALENT NOISE IN A FIRST-ORDER CORRELATION LOOP UTILIZING FILTERED PSEUDONOISE SIGNALS

Christopher J. Scholten

Dynetics, Inc.†
Huntsville, AL 35806

and

Kurt L. Kosbar

University of Missouri - Rolla
Rolla, MO 65401

Abstract - Conventional direct-sequence and frequency-hopping spread spectrum systems utilize delay-lock loops to track the timing epoch of pseudonoise codes. These devices perform admirably at high signal-to-noise ratios (SNR), however, they are suboptimal at moderate and low SNR. A correlation loop that employs appropriately filtered pseudonoise signals for the local cross-correlation waveform may have superior performance under these conditions. The performance of this modified correlation loop will be determined by the cross correlation function of the transmitted waveform and locally generated reference signals, along with the statistics of the baseband equivalent noise process. In this work, we find the statistics of the baseband equivalent noise process. The approach is reasonably general and can be applied to a variety of signal structures and pre-correlation filters. In many interesting cases, it is possible to use Central Limit Theorem arguments to show that the equivalent noise is approximately additive, white and Gaussian over the loop bandwidth.

I. Introduction

The phase-lock loop (PLL) [1] and delay-lock loop (DLL) [2] are two devices that have been extensively studied and widely employed [3]. These loops operate by multiplying a received signal by a locally generated reference waveform. This product is then filtered and used to control the delay of the local waveform generator. In both cases, the reference signal is approximately a differentiated replica of the transmitted waveform.

The PLL and DLL are optimal at high SNR but may be suboptimal when operating at low SNR. For this case we can consider the generalized correlation loop of Figure 1. The performance of this loop may be characterized by the three terms at the output of the multiplier: the discriminator characteristic, $R_{xl}(\varepsilon)$, the self-noise term, $b(t,\varepsilon)$, and the thermal noise term, $n'(t) = n(t)l(t,\varepsilon)$. Using these three quantities, a non-linear equivalent model for this loop may be developed, which in turn may be linearized to determine measures such as loop bandwidth and loop performance.

As mentioned above, $R_{xl}(\varepsilon)$ is known as the discriminator characteristic. This term is the cross-correlation function of $x(t)$ and $l(t,\varepsilon)$ or the DC component of $x(t)l(t,\varepsilon)$. This DC term is used for code tracking. The AC component of $x(t)l(t,\varepsilon)$, denoted as $b(t,\varepsilon)$, does not aid in tracking of the received code and as such it is a self-noise term since it is generated solely by the loop. Finally, $n'(t) = n(t)l(t,\varepsilon)$ is a term arising from multiplication of the received thermal noise by the local reference. Thus in order to characterize the loop of Figure 1 we must find $R_{xl}(\varepsilon)$ and the statistics of the baseband equivalent noise processes, $b(t,\varepsilon)$ and $n'(t)$.

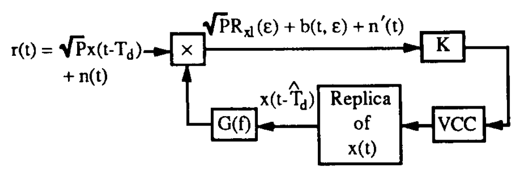

Figure 1. Correlation Loop

In the conventional PLL and DLL, the linear time-invariant filter, G(f), in Figure 1 is a differentiator. It is not, however, the differentiation operation specifically that introduces a useful error signal; instead, the resultant ninety degree phase shift from this operation generates the usable error term. Thus the loop will operate with other G(f), provided the phase response is ninety degrees over a wide range of frequencies. At low SNR, it is possible that an amplitude response other than that of a differentiator will be optimal.

In this paper we will examine the statistics of the baseband equivalent noise processes for arbitrary G(f) and plot the self-noise for the specific case where $|G(f)|=1$ for all frequencies; i. e. G(f) is a Hilbert transformer. Section II discusses the pseudonoise signal model used in this work. Self-noise components are evaluated in

† This work was completed while a student at the University of Missouri - Rolla.

This work was supported in part by Engineering Foundation Grant #RI-B-90-25.

CH2981-9/91/0000-0355 $1.00 © 1991 IEEE

Section III and the thermal noise is examined in Section IV. Finally, the results are summarized in Section V.

II. Signal Model

Suppose we wish to estimate the delay of a pseudonoise (PN) code [4], x(t), which is a periodic random process with sample functions that are described by (over one period)

$$x(t) = \sum_{n=0}^{\alpha} a_n p\left(\frac{t-\zeta-nT_c}{T_c}\right) \tag{1}$$

where p(t)=1 for $|t|<1/2$ and is zero for all other t. The amplitudes, a_n, are independent, identically distributed Bernoulli random variables at ±1. The period, T, is given by αT_c where α is very large and T_c denotes one chip time. Finally, ζ is a random variable that is uniformly distributed in the interval $(0,T_c)$.

Because x(t) is periodic, we may expand it in a random Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t} \quad \text{where } X_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}dt \tag{2}$$

The Fourier coefficients, X_k, are random variables. We wish to determine the spectral properties of X_k, thus we must determine (2). Substituting (1) into (2) and performing the integration, it may be shown that X_k is given by

$$X_k = \frac{2\sin\left(k\omega_o\frac{T_c}{2}\right)}{k\omega_o T}[R_k+jI_k] \tag{3}$$

where $$R_k = \sum_{n=0}^{\alpha} a_n\cos(k\omega_o(nT_c+\zeta))$$

and $$I_k = -\sum_{n=0}^{\alpha} a_n\sin(k\omega_o(nT_c+\zeta)).$$

Since a_n and a_m are independent for $n\neq m$ and α is very large, R_k and I_k are approximately Gaussian by the Central Limit Theorem. It may also be shown that R_k and I_k are approximately uncorrelated for all k as α increases to infinity by determining the correlation coefficient, ρ, of R_k and I_k. Thus R_k and I_k are marginally Gaussian and uncorrelated. Since we can show that R_k and I_k are also jointly Gaussian [5], they are independent.

Using the above information, we may transform the random variable R_k+jI_k to a magnitude and phase, $Z_k\angle W_k$, where Z_k is a Rayleigh random variable (r.v.) and W_k is a uniform r.v.. Thus from (3), the magnitude of X_k is a Rayleigh distributed random variable weighted by a sinc function and the phase of X_k is uniform. After some algebraic manipulation, one can show that the phases of X_k and X_i are independent for $k\neq i$.

Thus for a periodic PN code stochastic process, x(t), the magnitude of its spectrum falls off as a sinc function. We also know that the phases of the spectral components are essentially independent from one harmonic to the next and uniformly distributed in the interval $(-\pi,\pi)$. These properties hold in the limit as the number of chips (α) per period goes off to infinity, which allows application of the Central Limit Theorem. The signal model used in the analysis of the correlation loop will be based on these results.

This approach is useful when signals other than "strict" PN codes are used which have similar properties. It is interesting to study cases where the signal spectrum decays as f^{-n} where $n\geq 1$. Thus we can overbound the magnitude spectrum by a deterministic envelope that is constant to some break frequency and then decays as f^{-n}. The signal model will also be periodic with harmonics having phases that are independent and uniformly distributed in $(-\pi,\pi)$.

III. Self-Noise

Referring to Figure 1, only the DC portion of $x(t)l(t,\varepsilon)$, $R_{xl}(\varepsilon)$, is used for tracking, thus the time-varying component is self-noise. Because x(t) and l(t) are periodic, they may be expanded as an exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{P}X_n}{\sqrt{P_x}} e^{jn\omega_o t} \tag{4}$$

$$l(t) = \sum_{n=-\infty}^{\infty} \frac{L_n}{\sqrt{P_l}} e^{jn\omega_o t} \tag{5}$$

where X_n is determined by (2) and L_n is found in a similar manner. The transmitted signal, x(t), has been normalized to a power of P Watts and the local reference to a power of unity where

$$P_l = \sum_{n=-\infty}^{\infty} |L_n|^2 \quad \text{and} \quad P_x = \sum_{n=-\infty}^{\infty} |X_n|^2.$$

The Fourier transform of the product, B(f), is given by the convolution X(f)*L(f) where X(f) and L(f) are the Fourier transforms of x(t) and l(t), respectively.

Performing the convolution yields

$$B(f) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\sqrt{P}X_n L_m}{\sqrt{P_x P_l}} \delta(f-(m+n)f_o). \quad (6)$$

Since $l(t) = g(t)*x(t-\varepsilon T)$ where g(t) is the impulse response of a linear, time-invariant filter that phase shifts all spectral components by ninety degrees, we have

$$|L_m| = |G(mf_o)||X_m| = |G_m||X_m| \quad (7)$$

$$\angle L_m = \angle G(mf_o) + \angle X_m - 2\pi m f_o \varepsilon T = -\frac{\pi}{2} + \theta_m - 2\pi m\varepsilon \quad (8)$$

where ε is the time delay in fractions of a period and θ_m is $\angle X_m$. Using (7) and (8) and letting m+n=k, (6) becomes

$$B_k = B(kf_o) = \quad (9)$$

$$\sum_{m=-\infty}^{\infty} \frac{\sqrt{P}|X_{-m+k}||G_m||X_m|}{\sqrt{P_x P_l}} e^{j(\theta_{-m+k}+\theta_m-\frac{\pi}{2}-2\pi m\varepsilon)} \delta(f-kf_o).$$

After some tedious manipulation, we may expand (9) into real and imaginary components.

$$\Re e(B_k) = \quad (10)$$

$$\sum_{m=\zeta(k)}^{\infty} [c_m \sin(\theta_{-m+k}+\theta_m) - d_m \cos(\theta_m+\theta_{-m+k})] \delta(f-kf_o)$$

$$\Im m(B_k) = \quad (11)$$

$$\sum_{m=\zeta(k)}^{\infty} [-d_m \sin(\theta_{-m+k}+\theta_m) - c_m \cos(\theta_m+\theta_{-m+k})] \delta(f-kf_o)$$

where

$$\zeta(k) = \frac{k}{2}, \text{ k even and } \frac{k+1}{2}, \text{ k odd}$$

$$c_m = a_m \cos(2\pi m\varepsilon) + b_m \cos(2\pi(k-m)\varepsilon)$$
$$d_m = a_m \sin(2\pi m\varepsilon) + b_m \sin(2\pi(k-m)\varepsilon)$$

$$a_m = \frac{\sqrt{P}|X_{-m+k}||G_m||X_m|}{\sqrt{P_x P_l}}, \ m \neq \frac{k}{2}$$

$$b_m = \frac{\sqrt{P}|X_m||G_{-m+k}||X_{-m+k}|}{\sqrt{P_x P_l}}, \ m \neq \frac{k}{2}$$

$$a_{k/2} = b_{k/2} = \frac{\sqrt{P}|X_{k/2}|^2|G_{k/2}|}{2\sqrt{P_x P_l}}, \ m = \frac{k}{2}$$

Each term in the sum over m is now independent. Denoting $\gamma_m = (\theta_{-m+k}+\theta_m)\text{modulo-}2\pi$, it is easy to show that the pdf of γ_m is uniform over $(-\pi,\pi)$. Letting

$$V_m = \sin\gamma_m \quad (12)$$
$$Y_m = \cos\gamma_m \quad (13)$$

(10) and (11) become

$$\Re e(B_k) = \sum_{m=\zeta(k)}^{\infty} [c_m V_m - d_m Y_m] \delta(f-kf_o) \quad (14)$$

$$\Im m(Bk) = \sum_{m=\zeta(k)}^{\infty} [-d_m V_m - c_m Y_m] \delta(f-kf_o). \quad (15)$$

The pdfs of (12) and (13) are given by

$$f_{V_m}(v) = f_{Y_m}(v) = \frac{1}{\pi\sqrt{1-v^2}}, \ |v|<1 \text{ and } 0, \text{ otherwise.} \quad (16)$$

Using (16), it is easy to show that (14) and (15) are zero mean and their variance is given by

$$\sigma^2(k) = E\,[\Re e(B_k)^2] = E\,[\Im m(B_k)^2] =$$

$$\sum_{m=\zeta(k)}^{\infty} \frac{a_m^2 + b_m^2 + 2a_m b_m E\,[\cos(2\pi(2m-k)\varepsilon)]}{2}$$

where $E\,[\cdot]$ denotes statistical expectation. The evaluation of the expected value over ε requires knowledge of the phase error pdf, $f_\varepsilon(\varepsilon)$, which can be difficult to obtain. Thus an overbound will be used. The bound is tight at high SNR, where the loop tracks with small phase errors. Thus

$$\sigma^2(k) \le \sum_{m=\zeta(k)}^{\infty} \frac{(a_m+b_m)^2}{2}. \quad (17)$$

Using arguments similar to those used to develop the signal model, $\Re e(B_k)$ and $\Im m(B_k)$ are Gaussian and independent. Transforming to $Z_k \angle W_k$ where Z_k is the magnitude of B_k and W_k is the phase of B_k, it is easy to show that Z_k is Rayleigh and W_k is uniform. It is tedious but not difficult to show that Z_k and Z_j are independent for $k \neq j$ and likewise W_k and W_j are independent.

To determine the statistics of the self-noise, we will examine the power spectral density of $b(t,\varepsilon)$. Assuming $b(t,\varepsilon)$ is ergodic, the power in $b(t,\varepsilon)$ is the time average of the energy or

$$E\left\{\frac{1}{T}\int_0^T |b(t,\varepsilon)|^2\,dt\right\}$$

$$= \sum_{n=-\infty}^{\infty} E\left\{|B_n|^2\right\} = \sum_{n=-\infty}^{\infty} 2\sigma^2(n)$$

The power spectral density is given by

$$S_b(f) = \sum_{k=-\infty}^{\infty} 2\sigma^2(k)\delta(f-kf_o) \qquad (18)$$

where $\sigma^2(k)$ is given by (17). Calculation of the power spectral density at each harmonic requires evaluation of (17) which contains an infinite sum. Finite sums will be used to generate an approximate result. The previous argument assumed (14) and (15) contained an infinite number of terms which allowed application of the Central Limit Theorem and the resulting conclusion that (14) and (15) were Gaussian. For finite sums they will not be Gaussian but for a large number of terms they should be approximately Gaussian and we can apply the previous results. Thus we must consider

$$\Re e(B_k) = \sum_{m=\zeta(k)}^{L} [c_m V_m - d_m Y_m]\delta(f-kf_o) \qquad (19)$$

$$\Im m(B_k) = \sum_{m=\zeta(k)}^{L} [-d_m V_m - c_m Y_m]\delta(f-kf_o). \qquad (20)$$

To check the Gaussian approximation, the first four moments of (19) and (20) were compared to the first four moments of a Gaussian density. Using the signal model of Figure 2 and $|G_m|=1$ for all frequencies, it was found that if $L\geq 500$, then the first four moments of (19) and (20) are within 1% of the first four moments of a Gaussian density. Thus if $L\geq 500$, we will consider (19) and (20) to be approximately Gaussian and the results obtained previously may be utilized. Thus the power spectral density becomes

$$S_b(f) = \sum_{k=-2L}^{2L} 2\sigma^2(k)\delta(f-kf_o) \qquad (21)$$

where $$\sigma^2(k) \leq \sum_{m=\zeta(k)}^{L} \frac{(a_m+b_m)^2}{2}. \qquad (22)$$

Equation (21) was calculated for the signal model in Figure 2 using $|G_m|=1$ and is illustrated in Figure 3 for various values of L. As can be seen, $S_b(kf_o)$ is reasonably flat for small k and, as L is increased, large numbers of harmonics are "packed" into the loop bandwidth. Thus as L is increased to infinity, $S_b(f)$ roughly approaches a continuous white Gaussian noise process within the loop bandwidth.

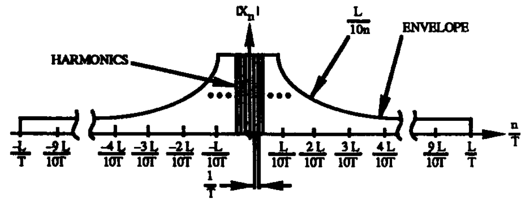

Figure 2. Magnitude Spectrum of x(t) for Finite L

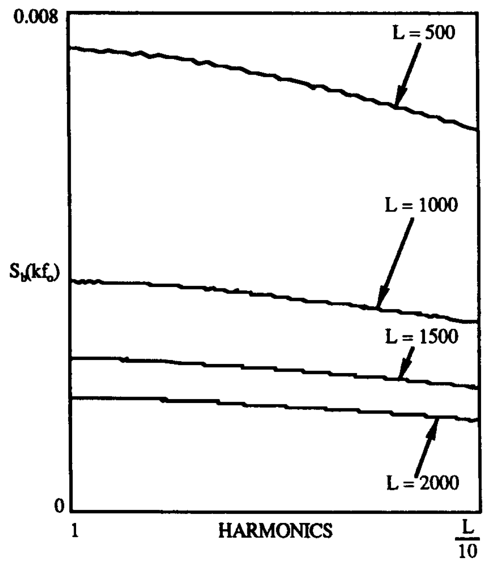

Figure 3. Self-Noise Power Spectral Density for Finite L

IV. Thermal Noise

The thermal noise term is due to the multiplication of the input noise by the local reference. Denoting the thermal noise by $n'(t)$,

$n'(t) = n(t)l(t,\varepsilon) =$

$$n(t) \sum_{m=-\infty}^{\infty} \frac{|X_m||G_m|}{\sqrt{P_l}} e^{j(m\omega_o t - 2\pi m\varepsilon - \frac{\pi}{2} + \theta_m)} \qquad (23)$$

where (5), (7) and (8) have been utilized. Denoting the magnitude $\frac{|X_m||G_m|}{\sqrt{P_l}}$ by ξ_m and expanding (23) into real and imaginary components, we get

$$n'(t) = n(t) \times$$
$$\sum_{m=-\infty}^{\infty} \xi_m [\sin(m\omega_o t - 2\pi m\varepsilon + \theta_m) - j\cos(m\omega_o t - 2\pi m\varepsilon + \theta_m)]$$

Since n(t) is a Gaussian stochastic process, we can see that n'(t) is also Gaussian.

In order to determine the power spectral density of n'(t), the autocorrelation function must first be found.

$$R_{n'}(t_1,t_2) = E\,[n'(t_1)n'^*(t_2)]$$

or $R_{n'}(t_1,t_2) =$

$$\sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \xi_m \xi_i R_n(t_1-t_2) \times$$

$$E\,[\cos(m\omega_o t_1 - i\omega_o t_2 - 2\pi m\varepsilon + 2\pi i\varepsilon + \theta_m - \theta_i) + j\sin(m\omega_o t_1 - i\omega_o t_2 - 2\pi m\varepsilon + 2\pi i\varepsilon + \theta_m - \theta_i)] \qquad (24)$$

There are three cases to consider in (24): i=m, i=−m and $|i| \neq |m|$. By using the properties $\xi_{-m}=\xi_m$ and $\theta_{-m}=-\theta_m$ and by expanding the expected values in (24) into expected values over ε and θ_m, it is easy to show that $R_{n'}(t_1,t_2) = 0$ for i=−m and $|i| \neq |m|$. For i=m

$$R_{n'}(t_1,t_2) = R_{n'}(\tau) = \sum_{m=-\infty}^{\infty} \xi_m^2 \frac{N_o}{2}\delta(\tau) \qquad (25)$$

since $R_n(\tau) = \frac{N_o}{2}\delta(\tau)$. The sum in (25) becomes

$$\sum_{m=-\infty}^{\infty} \xi_m^2 = \sum_{m=-\infty}^{\infty} \frac{|X_m|^2|G_m|^2}{P_l} = \frac{P_l}{P_l} = 1$$

so that (25) reduces to

$$R_{n'}(\tau) = \frac{N_o}{2}\delta(\tau).$$

Thus n'(t) is white Gaussian noise with a double-sided power spectral density of $\frac{N_o W}{2\ Hz}$ at all frequencies. If l(t,ε) had not been normalized to unit power and instead had power P_l, the power spectral density would be given by $\frac{P_l N_o W}{2\ Hz}$.

In general, n'(t) would not be stationary; however, the uniformly distributed phases of the signal model perform a "phase randomizing" of the thermal noise at the multiplier output which makes n'(t) stationary. The signal model properties in turn relied on the application of the Central Limit Theorem and became valid as the number of chips per period increased to infinity.

V. Conclusions

In this paper, a method was presented for determining the statistics of the self-noise in a first-order correlation loop using signals with pseudonoise type properties. A model for this type of signal was developed. In addition, the statistics of the thermal noise in the loop were examined.

It was determined that, for the signal model used and with $|G_m|=1$ at all frequencies, the code self-noise was periodic and approached a bandlimited, white Gaussian noise process over the loop bandwidth as the period of the signal model approached infinity. This result holds as the number of chips per period increases to infinity which allows application of the Central Limit Theorem.

It was also found that the thermal noise at the multiplier output was white and Gaussian. This occurred because the uniformly distributed phases of the signal model harmonics acted to "phase randomize" the multiplier output.

VI. References

[1] Viterbi, A. J., "Phase-Lock Loop Dynamics in the Presence of Noise by Fokker-Planck Techniques, Proceedings of the IEEE, Vol. 51, No. 12, pp. 1737-1753, December 1963.

[2] Spilker, J. J., "Delay-Lock Tracking of Binary Signals", IEEE Transactions on Space Electronics and Telemetry, Vol. SET-9, pp. 1-8, March 1963.

[3] Lindsey, W. C. and Simon, M. K. et al., Phase-Locked Loops and Their Applications. New York: IEEE Press, 1978.

[4] Ziemer, R. E. and Peterson, R. L., Digital Communications and Spread Spectrum Systems, Chapter 8. New York: Macmillan, 1985.

[5] Scholten, C. J., "Analysis of Baseband Equivalent Noise in a First-Order Correlation Loop Utilizing Pseudonoise Signals", Master's Thesis, University of Missouri - Rolla, 1991.